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Abstract

We show that a composition operator on weighted Bergman spaces $\mathcal{A}^p_\mu$ is invertible if and only if it is Fredholm if and only if it is an isometry.

1 Introduction

In [8], the authors defined the Hardy space $\mathcal{H}^2$ of Dirichlet series with square-summable coefficients. Thanks to the Cauchy-Schwarz inequality, it is easy to see that $\mathcal{H}^2$ is a space of analytic functions on $\mathbb{C}_{\frac{1}{2}} := \{s \in \mathbb{C},\, \Re(s) > \frac{1}{2}\}$. F. Bayart introduced in [3] the more general class of Hardy spaces of Dirichlet series $\mathcal{H}^p$ $(1 \leq p < +\infty)$. In another direction, McCarthy defined in [12] some weighted Bergman Hilbert spaces of Dirichlet series and these spaces have been generalized in [2].

In order to recall how these spaces are defined, we need to recall the principle of the Bohr's point of view: let $n \geq 2$ be an integer, it can be written (in a unique way) as a product of prime numbers $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ where $p_1 = 2$, $p_2 = 3$ etc ... For $s \in \mathbb{C}$, we consider $z = (z_1, z_2, \dots) = (p_1^{-s}, p_2^{-s}, \dots)$. Then, writing

$$f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} \tag{1}$$

we get

$$f(s) = \sum_{n=1}^{+\infty} a_n (p_1^{-s})^{\alpha_1} \cdots (p_k^{-s})^{\alpha_k} = \sum_{n=1}^{+\infty} a_n\, z_1^{\alpha_1} \cdots z_k^{\alpha_k}.$$

So we can see a Dirichlet series as a Fourier series on the infinite-dimensional polytorus $\mathbb{T}^\infty = \{(z_1, z_2, \cdots),\, |z_i| = 1,\, \forall i \geq 1\}$. We shall denote this Fourier series $D(f)$.

Let us fix now $p \geq 1$. The space $H^p(\mathbb{T}^\infty)$ is the closure of the set of analytic polynomials with respect to the norm of $L^p(\mathbb{T}^\infty, m)$ where m is the normalized Lebesgue measure on $\mathbb{T}^\infty$. Let f be a Dirichlet polynomial, $D(f)$ is then an analytic polynomial on $\mathbb{T}^\infty$ by the Bohr's point of view. By definition, $\|f\|_{\mathcal{H}^p} := \|D(f)\|_{H^p(\mathbb{T}^\infty)}$ and $\mathcal{H}^p$ is the closure of the set of Dirichlet polynomials with respect to this norm. The spaces $\mathcal{H}^p$ and $H^p(\mathbb{T}^\infty)$ are then isometrically isomorphic.

We recall now how we can define the weighted Bergman spaces of Dirichlet series. For $\sigma > 0$, f_σ will be the translate of f by σ, *i.e.* $f_\sigma(s) := f(\sigma + s)$. We shall denote by $\mathcal{P}$ the set of Dirichlet polynomials.

Let $p \geq 1$, $P \in \mathcal{P}$ and μ be a probability measure on $(0, +\infty)$ such that $0 \in Supp(\mu)$. Then

$$\|P\|_{\mathcal{A}^p_\mu} := \left(\int_0^{+\infty} \|P_\sigma\|^p_{\mathcal{H}^p}\, d\mu(\sigma) \right)^{1/p}.$$

$\mathcal{A}^p_\mu$ is the completion of $\mathcal{P}$ with respect to this norm. When $d\mu(\sigma) = 2e^{-2\sigma}\, d\sigma$, these spaces are simply denoted by $\mathcal{A}^p$. It is shown in [2] that they are spaces of convergent Dirichlet series on $\mathbb{C}_{1/2}$.

In [7], the bounded composition operators on $\mathcal{H}^2$, in other words the analytic functions $\Phi : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ such that for any $f \in \mathcal{H}^2$, $f \circ \Phi \in \mathcal{H}^2$, are characterized. In [3], F. Bayart generalized this result to the space $\mathcal{H}^p$ when $p \geq 1$.

We denote by $\mathcal{D}$ the set of functions f which admit a representation by a convergent Dirichlet series in some half-plane and for $\theta \in \mathbb{R}$, $\mathbb{C}_\theta$ will be the following half-plane $\{s \in \mathbb{C},\, \Re(s) > \theta\}$. We shall denote $\mathbb{C}_+$ instead of $\mathbb{C}_0$.

On the spaces $\mathcal{A}^p_\mu$, the following theorems have been proved in [1]:

Theorem 1 ([1],Th1). *Let $\Phi : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ be an analytic function of the form $\Phi(s) = c_0 s + \varphi(s)$ where $c_0 \geq 1$ and $\varphi \in \mathcal{D}$. Then C_Φ is bounded on $\mathcal{A}^p_\mu$ if and only if φ converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$. Moreover in this case, C_Φ is a contraction.*

Theorem 2 ([1],Th2). *Let $\Phi : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ be in $\mathcal{D}$. Then*

(i) If C_Φ is bounded on $\mathcal{A}^p_\mu$ then Φ converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and $\Phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$.

(ii) If Φ converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and $\Phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2+\eta}$ with some $\eta > 0$ then C_Φ is bounded on $\mathcal{A}^2_\mu$.

In the sequel, we assume that μ is a probability measure on $(0, +\infty)$ such that $d\mu(\sigma) = h(\sigma)d\sigma$ where h is a positive continuous function.

Example. Let $\alpha > -1$, we denote μ_α the probability measure defined on $(0, +\infty)$ by

$$d\mu_\alpha(\sigma) = \frac{2^{\alpha+1}}{\Gamma(\alpha+1)}\sigma^\alpha e^{-2\sigma}\, d\sigma.$$

We denote the corresponding space $\mathcal{A}^p_\alpha$ instead of $\mathcal{A}^p_{\mu_\alpha}$.

Main Theorem. *Let $1 \leq p < +\infty$ and C_Φ be a bounded composition operator on $\mathcal{A}^p_\mu$. The following assertions are equivalent:*

(i) C_Φ is inversible.

(ii) C_Φ is Fredholm.

(iii) C_Φ is an isometry.

(iv) Φ is a vertical translation: there exists $\tau \in \mathbb{R}$ such that for every $s \in \mathbb{C}_+$, $\Phi(s) = s + i\tau$.

We point out that the result is false on the spaces $\mathcal{H}^p$: F. Bayart proved that $(i), (ii), (iii)$ are still equivalent on $\mathcal{H}^p$ but obtained a different characterization for the isometric composition operators on $\mathcal{H}^p$ (see [3]). For example, if Φ is defined for every $s \in \mathbb{C}_+$ by $\Phi(s) = c_0 s$ with $c_0 \geq 2$, then C_Φ is an isometry on $\mathcal{H}^p$ but not on $\mathcal{A}^p_\mu$. The same phenomenon appears in the framework of composition operators on the unit disk (see [10]).

Ir order to prove the main theorem, it suffices to show that $(ii) \Rightarrow (iv)$ and $(iii) \Rightarrow (iv)$. Indeed, $(i) \Rightarrow (ii)$, $(iv) \Rightarrow (i)$ and $(iv) \Rightarrow (iii)$ are clear.

2 Background material

Let f be a Dirichlet series of form (1). We do not recall the definition of abscissa of simple (resp. absolute) convergence denoted by σ_c (resp. σ_a), see [13] or [14] for more details. We shall need the two other following abscissas:

$$\begin{aligned}\sigma_u(f) &= \inf\{a \mid \text{The series (1) is uniformly convergent for } \Re(s) > a\} \\ &= \text{abscissa of uniform convergence of } f.\end{aligned}$$

$$\begin{aligned}\sigma_b(f) &= \inf\{a \mid \text{the function } f \text{ has an analytic, bounded extension for } \Re(s) > a\} \\ &= \text{abscissa of boundedness of } f.\end{aligned}$$

It is easy to see that $\sigma_c(f) \leq \sigma_u(f) \leq \sigma_a(f)$. An important result is that $\sigma_u(f)$ and $\sigma_b(f)$ coincide: this is the Bohr's theorem (see [5]). This result is really useful for the study of $\mathcal{H}^\infty$, the algebra of bounded Dirichlet series on the right half-plane $\mathbb{C}_+$ (see [11]). We shall denote by $\|\cdot\|_\infty$ the norm on this space:

$$\|f\|_\infty := \sup_{\Re(s)>0} |f(s)|.$$

We shall make a crucial use of the point evaluation in the proof of the Main Theorem: for every $p \geq 1$, the spaces $\mathcal{H}^p$ and $\mathcal{A}^p_\mu$ are spaces of holomorphic functions on $\mathbb{C}_{1/2}$ and more precisely if δ_s is the operator of point evaluation at $s \in \mathbb{C}_{1/2}$, then by [3],Th3:

$$\|\delta_s\|_{(\mathcal{H}^p)^*} = \zeta(2\Re(s))$$

and by [2],Th1 the point evaluation is also bounded on the spaces $\mathcal{A}^p_\mu$. Moreover $\sigma_b(f) \leq 1/2$ for any $f \in \mathcal{A}^p_\mu$. For example when $\mu = \mu_\alpha$, it is shown in [2],Cor1 that there exists a positive constant $c_{\alpha,p}$ such that for every $s \in \mathbb{C}_{1/2}$,

$$\|\delta_s\|_{(\mathcal{A}^p_\alpha)^*} \leq c_{\alpha,p}\left(\frac{\Re(s)}{2\Re(s)-1}\right)^{\frac{2+\alpha}{p}}.$$

When $p = 2$, $\mathcal{A}^2_\mu$ is a Hilbert space and it is easy to see that

$$\|f\|_{\mathcal{A}^2_\mu} = \left(\sum_{n=1}^{+\infty} |a_n|^2 w_h(n)\right)^{1/2}$$

where for every $n \geq 1$,

$$w_h(n) = \int_0^{+\infty} n^{-2\sigma} h(\sigma) d\sigma.$$

Thanks of the boundedness of the point evaluation at $s \in \mathbb{C}_{1/2}$, we consider the following reproducing kernels defined for every $w \in \mathbb{C}_{1/2}$ by

$$K_\mu(s,w) = \sum_{n=1}^{+\infty} \frac{n^{-\bar{s}-w}}{w_h(n)}.$$

For every $f \in \mathcal{A}^2_\mu$ and $s \in \mathbb{C}_{1/2}$, one has

$$f(s) = < f, K_\mu(s,\cdot) >_{\mathcal{A}^2_\mu} \cdot$$

Example. On the space $\mathcal{A}^2_\alpha$, we simply denote (w_n^α) the corresponding weight and then for every $n \geq 1$,

$$w_n^\alpha = \frac{1}{(\log(n)+1)^{\alpha+1}}.$$

Let $\Phi : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ be an analytic function such that $\Phi(s) = c_0 s + \varphi(s)$ where c_0 is a nonnegative integer and $\varphi \in \mathcal{D}$. We shall say that Φ is a symbol if C_Φ is bounded on the spaces $\mathcal{A}^p_\mu$.

For $\sigma > 0$, we denote Φ_σ the translate of Φ by σ: $\Phi_\sigma(s) := \Phi(\sigma + s)$.

When $c_0 \geq 1$, thanks to the Theorem 1 we know that Φ is a symbol if and only φ converges uniformly on $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$. In this case, it is easy to see that for every $\sigma > 0$, $\Phi_\sigma - \sigma$ is also a symbol: indeed let $\sigma > 0$ and $s \in \mathbb{C}_+$, then

$$\begin{aligned} \Re(\Phi_\sigma(s) - \sigma) &= \Re(c_0(\sigma + s)) + \Re(\varphi(\sigma + s)) - \sigma \\ &> \sigma(c_0 - 1) + \Re(s) > 0 \end{aligned}$$

because $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$ and $s \in \mathbb{C}_+$. Point out that this result can be seen as the Schwarz's lemma in this framework.

3 Proof of $(ii) \Rightarrow (iv)$

With help of Proposition 4.2 from [7], F. Bayart proved the following useful lemma.

Lemma 1 ([3],Lem11)**.** *Let Φ be a symbol. If Φ is not a vertical translation then there exists ε and $\eta > 0$ such that*

$$\Phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}.$$

Proof of $(ii) \Rightarrow (iv)$. We follow ideas from [3],Th14. Assume that Φ is not a vertical translation. By the previous lemma, there exists ε and $\eta > 0$ such that

$$\Phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}.$$

We remark that each element of $\mathrm{Im}(C_\Phi)$ is defined and bounded on $\mathbb{C}_{1/2-\varepsilon}$: indeed $\Phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}$ and if $f \in \mathcal{A}^p_\mu$, f is bounded on $\mathbb{C}_{1/2+\eta}$ (because $\sigma_b(f) \leq 1/2$).

Now by lemma [3],Lem9 we know that there exists $f \in \mathcal{H}^p$ such that the line $\Re(s) = 1/2$ is both abscissa of convergence and natural boundary for f. Because of the inclusion $\mathcal{H}^p \subset \mathcal{A}^p_\mu$, f belongs to $\mathcal{A}^p_\mu$. We consider the following infinite dimensional subspace of $\mathcal{A}^p_\mu$:

$$F = \mathrm{span}\{n^{-s} f,\, n \geq 1\} = f\mathcal{P}.$$

We shall show that $F \cap Im(C_\Phi) = \{0\}$ and consequently $\mathrm{Codim}(\mathrm{Im}(C_\Phi))=+\infty$ which is a contradiction with (ii).

Let $h \in F \cap Im(C_\Phi)$, there exists $P \in \mathcal{P}$ such that $h = Pf$. If $h \neq 0$, there exists s_0 such that $\Re(s_0) = 1/2$ and $P(s_0) \neq 0$. But in this case, f extends beyond $\mathbb{C}_{1/2}$ and then we obtain a contradiction because the line $\Re(s) = 1/2$ is a natural boundary for f. Finally $F \cap Im(C_\Phi) = \{0\}$. $\square$

4 Proof of $(iii) \Rightarrow (iv)$

First we shall show that if C_Φ is an isometry then $c_0 \geq 1$. We need the following result.

Lemma 2. $\lim_{\Re(s)\to+\infty} \|\delta_s\|_{(\mathcal{A}^1_\mu)^*} = 1.$

Proof. Let $s \in \mathbb{C}_1$. By the reproducing kernel property on $\mathcal{A}^2_\mu$ (or just by a simple computation), for any Dirichlet polynomial we have

$$P(s) = \int_0^{+\infty} \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} P(\sigma+it)\overline{K_\mu(s,\sigma+it)}\, dt d\mu(\sigma).$$

Now by definition of the norm of Dirichlet polynomials in $\mathcal{H}^1$ (see definition 1 from [3]), we have

$$\|P\|_{\mathcal{A}^1_\mu} = \left(\int_0^{+\infty} \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |P(\sigma+it)| dt d\mu(\sigma) \right).$$

Consequently

$$|P(s)| \leq \|P\|_{\mathcal{A}^1_\mu} \times \|K_\mu(s,\cdot)\|_\infty.$$

Now

$$\|K_\mu(s,\cdot)\|_\infty = \sup_{w\in\mathbb{C}_+} \left| \sum_{n=1}^{+\infty} \frac{n^{-\overline{s}+w}}{w_h(n)} \right| \leq \sum_{n=1}^{+\infty} \frac{n^{-\Re(s)}}{w_h(n)}$$

and we point out that $w_h(1) = 1$ so

$$\limsup_{\Re(s)\to+\infty} \|\delta_s\|_{(\mathcal{A}^1_\mu)^*} \leq \lim_{\Re(s)\to+\infty} \sum_{n=1}^{+\infty} \frac{n^{-\Re(s)}}{w_h(n)} = 1.$$

On the other hand, it is clear that $\|\delta_s\|_{(\mathcal{A}^1_\mu)^*} \geq 1$ and then we obtain the result. $\square$

Proposition 1. *Let Φ be a symbol. If C_Φ is a contraction then $c_0 \geq 1$.*

Proof. Let $s \in \mathbb{C}_{1/2}$. For every $f \in \mathcal{A}^p_\mu$ we have

$$|f\circ\Phi(s)| \leq \|\delta_s\|_{(\mathcal{A}^p_\mu)^*}\|f\circ\Phi\| \leq \|\delta_s\|_{(\mathcal{A}^p_\mu)^*}\|C_\Phi\|\|f\|$$

and then

$$\frac{\|\delta_{\Phi(s)}\|_{(\mathcal{A}^p_\mu)^*}}{\|\delta_s\|_{(\mathcal{A}^p_\mu)^*}} \leq \|C_\Phi\|.$$

By inclusion of the spaces $\mathcal{A}^p_\mu$ and the fact that $\mathcal{H}^p \subset \mathcal{A}^p_\mu$ with $\|\cdot\|_{\mathcal{A}^p_\mu} \leq \|\cdot\|_{\mathcal{H}^p}$ we obtain:

$$\frac{\|\delta_{\Phi(s)}\|_{(\mathcal{H}^p)^*}}{\|\delta_s\|_{(\mathcal{A}^1_\mu)^*}} \leq \|C_\Phi\|.$$

By Theorem 3 from [3], we obtain

$$\zeta(2\Re(\Phi(s)))^{1/p} \times \|\delta_s\|^{-1}_{(\mathcal{A}^1_\mu)^*} \leq \|C_\Phi\|.$$

Now assume $c_0 = 0$, then $\Phi(s) = \varphi(s) = \sum_{n=1}^{+\infty} c_n n^{-s}$ and $\Re(c_1) > 1/2$ (see proof of Lemma 3.3 from [7]). Finally thanks to the Lemma 2, when $\Re(s)$ goes to infinity we get

$$\|C_\Phi\| \geq \zeta(2\Re(c_1))^{1/p} > 1$$

and consequently C_Φ is not a contraction. $\square$

Remark. In the previous Lemma we actually used that for every $s \in \mathbb{C}_{1/2}$, $\delta_s \circ C_\Phi = \delta_{\Phi(s)}$.

Proof of (iii) $\Rightarrow$ (iv). Assume that C_Φ is an isometry. By the last lemma, $c_0 \geq 1$ and then we know that $\Phi : \mathbb{C}_+ \to \mathbb{C}_+$ thanks to the Theorem 1. One has

$$\|2^{-s}\|_{\mathcal{A}^p_\mu} = \|2^{-\Phi}\|_{\mathcal{A}^p_\mu}.$$

Now by [2],Th6,

$$\int_0^{+\infty} \|2^{-\sigma-\bullet}\|^p_{\mathcal{H}^p} - \|2^{-\Phi(\sigma+\bullet)}\|^p_{\mathcal{H}^p}\, d\mu(\sigma) = 0.$$

But

$$\|2^{-\Phi(\sigma+\bullet)}\|_{\mathcal{H}^p} = \|2^{-\sigma-(\Phi(\sigma+\bullet)-\sigma)}\|_{\mathcal{H}^p} = \|C_{\Phi_\sigma-\sigma}(2^{-\sigma-\bullet})\|_{\mathcal{H}^p}.$$

Thanks to the Schwarz's lemma in this framework (recall that $c_0 \geq 1$) we know that $\Phi_\sigma - \sigma : \mathbb{C}_+ \to \mathbb{C}_+$. So by the Theorem 1, $C_{\Phi_\sigma-\sigma}$ is a bounded composition operator on $\mathcal{H}^p$ and $\|C_{\Phi_\sigma-\sigma}\| \leq 1$. Then

$$\|2^{-\Phi(\sigma+\bullet)}\|_{\mathcal{H}^p} \leq \|2^{-\sigma-\bullet}\|_{\mathcal{H}^p}.$$

Consequently $2^{-\sigma} = \|2^{-\sigma-\bullet}\|_{\mathcal{H}^p} = \|2^{-\Phi(\sigma+\bullet)}\|_{\mathcal{H}^p}$ for every $\sigma > 0$ (recall that h is a positive continuous function). Now by Lemma 1, if Φ is not a vertical translation, there exists ε and $\eta > 0$ such that $\Phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}$ and then for every $\sigma > 1/2 - \varepsilon$,

$$2^{-\sigma} = \|2^{-\Phi(\sigma+\bullet)}\|_{\mathcal{H}^p} \leq \|2^{-\Phi(\sigma+\bullet)}\|_{\mathcal{H}^\infty} \leq 2^{-1/2-\eta}$$

and this is obviously false. $\square$

Remark. Let μ be a probability measure on $(0, +\infty)$ such that $0 \in Supp(\mu)$ and $d\mu = h d\sigma$ where h is a nonnegative function. If there exists an open interval I such that h is positive on I then the theorem still holds. It is a consequence of the following lemma and some easy adaptations of the previous proof.

Lemma 3. *Let Φ be a symbol with $c_0 \geq 1$. If Φ is not a vertical translation then for every $\varepsilon > 0$, there exists $\eta = \eta_\varepsilon > 0$ such that $\Phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_{\varepsilon+\eta}$.*

Proof. First we assume that φ is non constant then $\varphi : \mathbb{C}_+ \to \mathbb{C}_+$ and by Proposition 4.2 from [7], there exists $\vartheta > 0$ such that $\varphi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_\vartheta$ and consequently $\Phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_{c_0\varepsilon+\vartheta}$. In this case, it suffices to choose $\eta = (c_0 - 1)\varepsilon + \vartheta$ which is positive because $c_0 \geq 1$.

If φ is constant equals to $i\tau$ ($\tau \in \mathbb{R}$) and $c_0 > 1$ then $\Phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_{c_0\varepsilon}$ and it suffices to choose $\eta = (c_0 - 1)\varepsilon$.

If φ is constant and equals to $c_1 \in \mathbb{C}_+$ and $c_0 \geq 1$, $\Phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_{c_0\varepsilon+\Re(c_1)}$ and it suffices to choose $\eta = (c_0 - 1)\varepsilon + \Re(c_1)$. $\square$

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